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Continuity of Bargaining Solutions

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Abstract: Upper semicontinuous solutions of the bargaining problem are studied and also lower semicontinuous weak solutions of that problem are considered. Though mainly compact bargaining pairs are investigated, extensions to non-compact bargaining pairs are indicated. The continuity properties of some well known bargaining solutions are discussed.

1. Introduction and Summary

Bargaining problems arise in many situations. A great number of solutions for such problems have been proposed in the literature [*Nash, Raiffa, Harsanyi/Selten, Yu, Kalai/Smorodinsky, Kalai, Kalai/Rosenthal*] each with its own advantages and disadvantages [*Schmitz, Roth, Rauhut/Schmitz/Zachow*].

Most of the authors consider solutions only on the class $\mathcal{B}^0$ of bargaining problems where the disagreement point is strictly dominated by some outcome in the set of feasible utility pairs, which are attainable by cooperation. We are interested in this paper in solutions defined on the whole class $\mathcal{B}$ of bargaining pairs. Indeed, bargaining pairs, where the disagreement point may be an element of the weak Pareto boundary of the set of feasible utility pairs, arise naturally in the theory of arbitration games. Use of the results of this paper is made by the authors [1981, 1982].

In this paper continuity properties of solutions are central. In section 3, we concentrate on upper semicontinuous solutions. Almost all proposed solutions in the literature appear to be upper semicontinuous, and many of them are even continuous on an open and dense subclass of the set of bargaining pairs (theorem 3.1 and propositions 3.1, 3.2). In theorem 3.3, four conditions on bargaining pairs are given, each of them guaranteeing that each upper semicontinuous solution is, in fact, continuous in the corresponding bargaining pair. The same theorem shows that these conditions are exhaustive. In theorem 3.4, open subclasses of bargaining pairs are described, where the coordinate functions ϕ_1 and ϕ_2 of an upper semicontinuous solution ϕ are continuous. In section 4, lower semicontinuous weak solutions are discussed. The main results are contained

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in theorems 4.1–4.5. Finally, in section 5, extensions to non-compact bargaining pairs are indicated. In section 2, we give definitions, examples and some lemmas, which are needed in the sequel.

2. Preliminaries

In the following, the family of non-empty compact, convex subsets of $\mathbf{R}^2$ is denoted by $\mathcal{K}$. For each $S \in \mathcal{K}$, the *Pareto set* of S

$$\{p \in S; \text{ for each } s \in S \text{ with } s \geqslant p, \text{ we have } s = p\}$$

is denoted by $\mathcal{P}(S)$ and the *weak Pareto set* of S

$$\{w \in S; \text{ for each } s \in S \text{ with } s \geqslant w, \text{ we have } s_1 = w_1 \text{ or } s_2 = w_2\}$$

is denoted by $\mathcal{W}(S)$.

We consider pairs (a, S), where $S \in \mathcal{K}$ and $a \in S$. Such pairs are called *bargaining pairs*. The set of all bargaining pairs is denoted by $\mathcal{B}$. An element $(a, S) \in \mathcal{B}$ corresponds, intuitively, to a situation, where two players are involved and where the i-th coordinate a_i of a is the level of utility that player i receives if they do not cooperate with each other, while S contains all the attainable points when they cooperate. Of course, for $s \in S$, s_i is the utility of outcome s for player $i \in \{1, 2\}$.

A map $\phi: \mathcal{B} \to \mathbf{R}^2$ will be called a *bargaining solution* or, shortly, a *solution*, if it satisfies the following properties:

(B.1) $\phi(a, S) \in \mathcal{P}(S)$, for each $(a, S) \in \mathcal{B}$ *(Pareto optimality)*,

(B.2) $\phi(a, S) \geqslant a$, for each $(a, S) \in \mathcal{B}$ (*Individual rationality*).

In the literature many solutions are discussed, and most of them have nice additional properties. We mention some of those solutions.

1. For $t \in (0, 1)$, the *generalized Nash solution* $\phi^{N(t)}: \mathcal{B} \to \mathbf{R}^2$ assigns to $(a, S) \in \mathcal{B}$

   the unique point $\phi^{N(t)}(a, S)$ of $\mathcal{P}(S)$, for which

$$(\phi_1^{N(t)}(a, S) - a_1)^t \, (\phi_2^{N(t)}(a, S) - a_2)^{1-t} = \max_{p \in \mathcal{P}(S)} (p_1 - a_1)^t \, (p_2 - a_2)^{1-t}.$$

   For $t = 1/2$ we have the Nash solution, introduced in *Nash* [1950], which has a nice axiomatic characterization. For more information on the solutions $\phi^{N(t)}$ we refer to *Harsanyi/Selten* [1972], *Kalai* [1977], *de Koster/Peters/Tijs/Wakker* [1983] and *Roth* [1979].

2. The *Kalai/Rosenthal* [1978] *solution* [see also *Raiffa*] $\phi^{KR}: \mathcal{B} \to \mathbf{R}^2$ assigns to $(a, S) \in \mathcal{B}$ the unique element in $[a, \mathcal{U}(S)] \cap \mathcal{P}(S)$. Here $\mathcal{U}(S) = (\max_{s \in S} s_1, \max_{s \in S} s_2)$ is the *utopia point* of S and $[a, \mathcal{U}(S)]$ the line segment with endpoints a and $\mathcal{U}(S)$.
3. The *Kalai/Smorodinsky* [1975] *solution* $\phi^{KS}: \mathcal{B} \to \mathbf{R}^2$ assigns to $(a, S) \in \mathcal{B}$ the unique element in $[a, \mathcal{U}(a, S)] \cap \mathcal{P}(S)$, where $\mathcal{U}(a, S)$ is the utopia point of $\{s \in S; s \geqslant a\}$. For an axiomatic characterization of this solution, we refer to *Kalai/Smorodinsky* [1975] and *Roth* [1979].
4. The *Yu* [1973] *(p)-solution* $\phi^{Y(p)}: \mathcal{B} \to \mathbf{R}^2$, for $p \in (1, \infty]$, assigns to $(a, S) \in \mathcal{B}$ the unique element in $\mathcal{P}(S)$ with minimal distance to $\mathcal{U}(a, S)$ in the norm $\| \; \|_p$, where $\| x \|_p = \sqrt[p]{|x_1|^p + |x_2|^p}$ if $p \in (1, \infty)$ and $\| x \|_\infty = \max \{|x_1|, |x_2|\}$.

   In the sequel also the following solutions play a role.
5. $\psi^1: \mathcal{B} \to \mathbf{R}^2$ and $\psi^2: \mathcal{B} \to \mathbf{R}^2$ are defined as follows. For $(a, S) \in \mathcal{B}$ and $i \in \{1, 2\}$, $\psi^i(a, S)$ is the element in $\{p \in \mathcal{P}(S); p \geqslant a\}$ with maximal i-th coordinate.

   In this paper we provide $\mathcal{K}$ with the Hausdorff metric $d_H: \mathcal{K} \times \mathcal{K} \to \mathbf{R}$, defined by

$$d_H(S, T) = \inf \{\epsilon > 0; S \subset B_\epsilon(T) \text{ and } T \subset B_\epsilon(S)\} \quad (S, T \in \mathcal{K})$$

where $B_\epsilon(S) = \{x \in \mathbf{R}^2; \inf_{s \in S} \| x - s \|_\infty \leqslant \epsilon\}$.

In the next two lemmas continuity properties are discussed of the multifunctions $\mathcal{W}: \mathcal{K} \to \mathbf{R}^2$ and $\mathcal{P}: \mathcal{K} \to \mathbf{R}^2$, which assign to each nonempty compact, convex subset of $\mathbf{R}^2$ its weak Pareto set $\mathcal{W}(S)$ and its Pareto set $\mathcal{P}(S)$, respectively. We recall the following

Definition. Let X, Y be metric spaces and let $F: X \to Y$ be a multifunction with compact image $F(x) \subset Y$ for each $x \in X$. We say that F is an *upper semicontinuous (u.s.c.) multifunction*, if for all sequences $(x, y), (x_1, y_1), (x_2, y_2), \ldots$ in $X \times Y$ with $\lim_{n \to \infty} (x_n, y_n) = (x, y)$ and $y_n \in F(x_n)$ for each $n \in \mathbf{N}$ we have: $y \in F(x)$. We say that F is a *lower semicontinuous (l.s.c.) multifunction* if the following holds: if for a sequence $x, x_1, x_2, x_3, \ldots$ in X with $\lim_{n \to \infty} x_n = x$ we have $y \in F(x)$, then there exists a sequence $y_1, y_2, \ldots$ with $y_n \in F(x_n)$ for each $n \in \mathbf{N}$ such that $\lim_{n \to \infty} y_n = y$.

Lemma 2.1. The multifunction $\mathcal{W}: \mathcal{K} \to \mathbf{R}^2$ is an u.s.c. multifunction.

Proof. Let $(S, w), (S_1, w_1), (S_2, w_2), \ldots$ be a sequence in $\mathcal{K} \times \mathbf{R}^2$ with $w_n \in \mathcal{W}(S_n)$ for each $n \in \mathbf{N}$ and $\lim_{n \to \infty} d_H(S, S_n) = 0$, $\lim_{n \to \infty} \| w_n - w \|_\infty = 0$. We have to prove that $w \in \mathcal{W}(S)$. Suppose $w \notin \mathcal{W}(S)$. Then there is an $s \in S$ with $s > w$.

Take a sequence $s_1, s_2, \ldots$ with $\lim_{n\to\infty} s_n = s$ and $s_n \in S_n$ for each $n \in \mathbf{N}$. Then, for n sufficiently large, we have $s_n > w_n$, which contradicts $w_n \in \mathcal{W}(S)$. Hence $w \in \mathcal{W}(S)$. Q.E.D.

Lemma 2.2. The multifunction $\mathcal{P}: \mathcal{K} \to \mathbf{R}^2$ *is l.s.c.*

Proof. Let $S, S_1, S_2, \ldots$ be a sequence in S with $\lim_{n\to\infty} d_H(S, S_n) = 0$ and let $p \in \mathcal{P}(S)$. We have to prove that there is a sequence $p_n \in \mathcal{P}(S_n)$ with $\lim_{n\to\infty} p_n = p$. Since $\lim_{n\to\infty} d_H(S, S_n) = 0$, there exists a sequence $s_1, s_2, \ldots$ with $\lim_{n\to\infty} s_n = p$ and $s_n \in S_n$ for each $n \in \mathbf{N}$. For each $n \in \mathbf{N}$ take $p_n \in \mathcal{P}(S_n)$ such that $p_n \geqslant s_n$. To prove that $\lim_{n\to\infty} p_n = p$ it is sufficient to show that each convergent subsequence of $p_1, p_2, \ldots$ converges to p. Let $p_{n(1)}, p_{n(2)}, p_{n(3)}, \ldots$ be a subsequence of $p_1, p_2, \ldots$ converging to q. Then $q \geqslant \lim_{k\to\infty} s_{n(k)} = p$. Since $q \in S$ and $p \in \mathcal{P}(S)$, we have $q = p$. Q.E.D.

We need also the following extension of the maximum theorem.

Lemma 2.3. Let X and Y be metric spaces. Let $F_1 : X \to Y$ *be a lower semicontinuous multifunction and* $F_2 : X \to Y$ *an upper semicontinuous multifunction such that* $F_1(x)$ *and* $F_2(x)$ *are compact sets and*

$$F_1(X) \subset F_2(x) \text{ for all } x \in X.$$

Let $f: X \times Y \to \mathbf{R}$ *be a continuous function with the property*

$$\max_{y\in F_1(x)} f(x, y) = \max_{y\in F_2(x)} f(x, y) \text{ for all } x \in X. \tag{2.1}$$

Let $M: X \to Y$ *be the multifunction defined by*

$$M(x) := \{\hat{y} \in F_2(x);\ f(x, \hat{y}) = \max_{y\in F_2(x)} f(x, y)\}.$$

Then M is an upper semicontinuous multifunction.

Proof. Let $(x^1, y^1), (x^2, y^2), \ldots$ be a sequence in $X \times Y$ with $y^n \in M(x^n)$ and $\lim_{n\to\infty} (x^n, y^n) = (\bar{x}, \bar{y})$. We have to show that $\bar{y} \in M(\bar{x})$. Since F_2 is u.s.c. and $M(x) \subset F_2(x)$ for all $x \in X$, we have $\bar{y} \in F_2(\bar{x})$. Suppose $\bar{y} \notin M(\bar{x})$. Then we can take in view of (2.1) an $y^* \in F_1(\bar{x})$ such that $2\epsilon := f(\bar{x}, y^*) - f(\bar{x}, \bar{y}) > 0$. In view of the lower semicontinuity of F_1 we can find a sequence $z^1, z^2, \ldots$ with $z^n \in F_1(x^n)$ for all $n \in \mathbf{N}$ such that $\lim_{n\to\infty} z^n = y^*$. Then

$$\lim_{n\to\infty} f(x^n, z^n) = f(\bar{x}, y^*) = f(\bar{x}, \bar{y}) + 2\epsilon = \lim_{n\to\infty} f(x^n, y^n) + 2\epsilon.$$

Hence, for n sufficiently large: $f(x^n, z^n) \geqslant f(x^n, y^n) + \epsilon$, which implies $y^n \notin M(x^n)$, a contraction. Hence, $\bar{y} \in M(\bar{x})$. Q.E.D.

3. Upper Semicontinuous Solutions

We provide $\mathcal{B}$ with the metric $d\colon \mathcal{B} \times \mathcal{B} \to \mathbf{R}$, defined by

$$d((a, S), (b, T)) = \max\{\|a - b\|_\infty, d_H(S, T)\}, \text{ for all } (a, S), (b, T) \in \mathcal{B}.$$

Note that the set $\mathcal{B}^0 = \{(a, S) \in \mathcal{B}; a \notin \mathcal{W}(S)\}$ is an open and dense subset of $\mathcal{B}$.

Remark. There does not exist a continuous solution. This can be shown as follows. For each $n \in \mathbf{N}$, let $a(n) = (1/n, 0)$ and $S_n = [(1/n, 0), (0, 1)]$. Then $\lim_{n\to\infty} (a(n), S_n) = (a, S)$, where $a = (0, 0)$ and $S = [(0, 0), (0, 1)]$. For each solution $\phi\colon \mathcal{B} \to \mathbf{R}^2$ we have

$\phi(a(n), S_n) = a(n)$, because $a(n) \in \mathcal{P}(S_n)$ and (B.1) and (B.2) hold

$\phi(a, S) = (0, 1)$, in view of (B.1) and the fact that $\mathcal{P}(S) = \{(0, 1)\}$.

Hence, $\lim_{n\to\infty} \phi(a(n), S_n) = (0, 0) \neq (0, 1) = \phi(a, S)$. So ϕ is not continuous. In fact we have proved that the function ϕ_2 is not lower semicontinuous in (a, S). So there are no lower semicontinuous solutions. Therefore we concentrate on upper semicontinuous solutions. We shall call a solution $\phi\colon \mathcal{B} \to \mathbf{R}^2$ *upper semicontinuous (u.s.c.)* if (B.3): for each sequence $(a, S), (a(1), S_1), (a(2), S_2), \ldots$ in $\mathcal{B}$ with

$\lim_{n\to\infty} (a(n), S_n) = (a, S)$, we have

$$\phi_i(a, S) \geqslant \limsup_{n\to\infty} \phi_i(a(n), S_n), \text{ for } i \in \{1, 2\},$$

or equivalently if

(B.3)′: for each $i \in \{1, 2\}$ and each $t \in \mathbf{R}$,

$$\phi_i^{-1}(-\infty, t) = \{(a, S) \in \mathcal{B}; \phi_i(a, S) < t\}$$

is an open subset of $\mathcal{B}$.

Theorem 3.1. Let $\phi: \mathcal{B} \to \mathbf{R}^2$ *be a bargaining solution. Then* ϕ *is u.s.c. in each point* $(a, S) \in \mathcal{B} \setminus \mathcal{B}^0$.

Proof. Suppose that ϕ is not u.s.c. in $(a, S) \in \mathcal{B} \setminus \mathcal{B}^0$.
Then there exists a sequence $(a(1), S_1), (a(2), S_2), \ldots$ in $\mathcal{B}$ with $\lim_{n\to\infty} (a(n), S_n) = (a, S)$, such that $\lim_{n\to\infty} \phi(a(n), S_n)$ exists and such that $\phi(a, S) \geqslant t := \lim_{n\to\infty} \phi(a(n), S_n)$ does not hold. Then either

(i) $\phi_1(a, S) < t_1$ **or** (ii) $\phi_2(a, S) < t_2$.

Without loss of generality, we suppose that (i) holds.

By lemma 2.1, $t \in \mathcal{W}(S)$. So $a_1 \leqslant \phi_1(a, S) < t_1$ implies that $\phi_2(a, S) = a_2 > t_2$. Let $\epsilon := a_2 - t_2 > 0$. Since $\lim_{n\to\infty} a(n) = a$, for n sufficiently large, we have

$$\phi_2(a(n), S_n) \geqslant a_2(n) > a_2 - 1/2\epsilon = t_2 + 1/2\epsilon.$$

Then $t_2 = \lim_{n\to\infty} \phi_2(a(n), S_n) \geqslant t_2 + 1/2\epsilon$ which is impossible. Q.E.D.

In the following proposition, we examine the upper semicontinuity of various solutions proposed in the literature.

Proposition 3.1. (i) *The following solutions are continuous on the class* $\mathcal{B}^0$: (a) *the Nash (t) solution* $\phi^{N(t)}$, (b) *the Kalai/Rosenthal solution* ϕ^{KR}, (c) *the Kalai/Smorodinsky solution* ϕ^{KS} *and* (d) *the Yu (p)-solution* $\phi^{Y(p)}$. (ii) *The solutions mentioned in* (i) *are upper semicontinuous on* $\mathcal{B}$.

Proof. (i) (a) If we apply lemma 2.3 with $X = \mathcal{B}^0$, $Y = \mathbf{R}^2$, $F_1: \mathcal{B}^0 \to \mathbf{R}^2$ is the l.s.c. multifunction defined by $F_1(a, S) = \mathcal{P}(\{x \in S; x \geqslant a\})$, $F_2: \mathcal{B}^0 \to \mathbf{R}^2$ is the u.s.c. multifunction defined by $F_2(a, S) = \mathcal{W}(\{x \in S; x \geqslant a\})$ and $f: \mathcal{B}^0 \times \mathbf{R}^2 \to \mathbf{R}$ is the continuous function defined by $f(a, S, y_1, y_2) = (y_1 - a_1)^t (y_2 - a_2)^{1-t}$ if $y \geqslant a$ and $f(a, S, y_1, y_2) = 0$ otherwise, then we find that the multifunction $M: \mathcal{B}^0 \to \mathbf{R}^2$ as defined in the same lemma is u.s.c. Since $M(a, S) = \{\phi^{N(t)}(a, S)\}$ this implies that $\phi^{N(t)}$ is a continuous function on $\mathcal{B}^0$.
(b) Let $\mu: \mathcal{B}^0 \to \mathbf{R}^2$ be the multifunction defined by $\mu(a, S) = \mathcal{W}(S) \cap [a, u(S)]$. Then $\mu(a, S) = \{\phi^{KR}(a, S)\}$. This implies that ϕ^{KR} is a continuous function on $\mathcal{B}^0$, since μ is u.s.c. because μ is the intersection of the u.s.c. multifunction $\mathcal{W}$ and the continuous multifunction $(a, S) \to [a, u(S)]$.
(c) Similarly, it follows that ϕ^{KS} is continuous on $\mathcal{B}^0$. We leave it to the reader to prove (d).
(ii) In view of theorem 3.1, (i) implies the upper semicontinuity on $\mathcal{B}$ of the solutions mentioned in the theorem. Q.E.D.

Proposition 3.2. The solutions ψ^1 and ψ^2 are upper semicontinuous. There are points in $\mathcal{B}^0$, where ψ^i is discontinuous.

Proof. We only look at ψ^2. That ψ_2^2 is u.s.c., follows from the fact that

$$\psi_2^2\,(a, S) = \max\,\{t;\, t \in \pi_2 \circ \lambda\,(a, S)\},$$

where λ is the u.s.c. multifunction with $\lambda\,(a, S) = \{w \in \mathcal{W}\,(S);\, w \geqslant a\}$ and π_i is defined by $\pi_i\,(x_1, x_2) = x_i$ for $i = 1, 2$. That ψ_1^2 is u.s.c., follows from the fact that

$$\psi_1^2\,(a, S) = \min\,\{t;\, t \in \pi_1 \circ \kappa\,(a, S)\},$$

where $\kappa\,(a, S) = \{p \in \mathcal{P}\,(S);\, p \geqslant a\}$ and κ is a l.s.c. multifunction by lemma 2.2.
To prove, for $i = 2$, the second assertion in the theorem, take $a\,(n) = (0, 0)$ and $S_n = \operatorname{conv}\,\{(0, 0), (0, 1), (1, (n-1)/n)\}$, for each $n \in \mathbf{N}$, $S = \operatorname{conv}\,\{(0, 0), (0, 1), (1, 1)\}$. Then $\lim\limits_{n\to\infty}\,(a\,(n), S_n) = ((0, 0), S)$ and $\lim\limits_{n\to\infty}\,\psi^2\,(a\,(n), S_n) = (0, 1) \neq (1, 1) = \psi^2\,(a, S)$ and obviously $(a, S) \in \mathcal{B}^0$. Q.E.D.

For an element $S \in \mathcal{K}$, we define

$$\bar{p}\,(S) = (\min_{p \in \mathcal{P}(S)} p_1,\ \max_{p \in \mathcal{P}(S)} p_2),\ \underline{p}\,(S) = (\max_{p \in \mathcal{P}(S)} p_1,\ \min_{p \in \mathcal{P}(S)} p_2),$$

$$\bar{\mathcal{W}}\,(S) = \{w \in \mathcal{W}\,(S);\, w_2 = \bar{p}_2\,(S)\},\ \underline{\mathcal{W}}\,(S) = \{w \in \mathcal{W}\,(S);\, w_1 = \underline{p}_1\,(S)\}.$$

Note, that $\mathcal{W}\,(S) = \bar{\mathcal{W}}\,(S) \cup \mathcal{P}\,(S) \cup \underline{\mathcal{W}}\,(S)$. Furthermore, let the *(jump-) function* $j_S : \mathcal{W}\,(S) \to \mathcal{P}\,(S)$ be defined as follows

$$j_S\,(w) = \begin{cases} \bar{p}\,(S) & w \in \bar{\mathcal{W}}\,(S) \\ w \quad \text{if} & w \in \mathcal{P}\,(S) \\ \underline{p}\,(S) & w \in \underline{\mathcal{W}}\,(S). \end{cases}$$

Theorem 3.2. Let $\phi\colon \mathcal{B} \to \mathbf{R}^2$ be an u.s.c. solution. Let (a, S), $(a\,(1), S_1)$, $(a\,(2), S_2), \ldots$ be a sequence in $\mathcal{B}$ with $\lim\limits_{n\to\infty}\,(a\,(n), S_n) = (a, S)$ and such that $\lim\limits_{n\to\infty} \phi(a(n), S_n)$ exists. Then $\lim\limits_{n\to\infty}\,\phi\,(a\,(n), S_n) \in \mathcal{W}\,(S)$ and $\phi\,(a, S) = j_S\,(\lim\limits_{n\to\infty}\,\phi\,(a\,(n), S_n))$.

Proof. It follows from lemma 2.1, that $\lim_{n\to\infty} \phi(a(n), S_n) \in \underline{W}(S)$. In view of the upper semicontinuity of ϕ_1 and ϕ_2, we have

$$\phi(a, S) \geqslant \lim_{n\to\infty} \phi(a(n), S_n) \tag{3.1}$$

Now we consider three cases:
If $\lim_{n\to\infty} \phi(a(n), S_n) \in P(S)$, then (3.1) and the fact that $\phi(a, S) \in P(S)$, imply that $\phi(a, S) = \lim_{n\to\infty} \phi(a(n), S_n)$.

If $\lim_{n\to\infty} \phi(a(n), S_n) \in \overline{W}(S)$, then (3.1) implies that $\phi(a, S) = \bar{p}(S)$, because $\phi(a, S) \in P(S)$ and because of the fact that $p = \bar{p}(S)$, for each $p \in P(S)$ with $p \geqslant w \in \overline{W}(S)$.

If $\lim_{n\to\infty} \phi(a(n), S_n) \in \underline{W}(S)$, then it follows, similarly, that $\phi(a, S) = \underline{p}(S)$.

Q.E.D.

Corollary 3.1. Let ϕ be an u.s.c. solution. Let $S \in K$ and let $a, a(1), a(2), \ldots$ be a sequence in S, such that $\lim_{n\to\infty} a(n) = a$.

Then $\lim_{n\to\infty} \phi(a(n), S) = \phi(a, S)$.

Proof. For each $n \in \mathbf{N}$, $\phi(a(n), S) \in P(S)$. Let z be a limit point of the sequence $\langle \phi(a(n), S); n \in \mathbf{N} \rangle$. Since $P(S)$ is closed, $z \in P(S)$ and, by theorem 3.2, $z = \phi(a, S)$. But then $\lim_{n\to\infty} \phi(a(n), S) = \phi(a, S)$. Q.E.D.

The following theorem describes four conditions on (a, S), each of them guaranteeing that each u.s.c. solution is continuous in (a, S). The same theorem also shows that these conditions are exhaustive.

Theorem 3.3. All u.s.c. solutions are continuous in (a, S) if and only if one of the following four conditions is satisfied:

(c.1) $P(S) = W(S)$,

(c.2) $a \geqslant (\bar{p}_1(S), \underline{p}_2(S))$,

(c.3) $a_1 \geqslant \bar{p}_1(S)$ *and* $\underline{W}(S) = \{\underline{p}(S)\}$,

(c.4) $a_2 \geqslant \underline{p}_2(S)$ *and* $\overline{W}(S) = \{\bar{p}(S)\}$.

Proof. (i) First we show that an u.s.c. solution $\phi: B \to \mathbf{R}^2$ is continuous in (a, S) if one of the conditions in the theorem is satisfied. Let $(a(1), S_1), (a(2), S_2), \ldots$ be a sequence in B, converging to (a, S). We have to prove that $\lim_{n\to\infty} \phi(a(n), S_n) = \phi(a, S)$.

Since the sequence $\phi(a(1), S_1), \phi(a(2), S_2), \ldots$ is bounded, it is sufficient to show that each convergent subsequence of this sequence converges to $\phi(a, S)$. Let $(a(n(1)), S_{n(1)}), (a(n(2)), S_{n(2)}), \ldots$ be such a convergent subsequence. Then, by theorem 3.2, $t = \lim_{k\to\infty} \phi(a(n(k)), S_{n(k)}) \in W(S)$ and $\phi(a, S) =$
$= j_S(\lim_{k\to\infty} \phi(a(n(k)), S_{n(k)}))$. We are finished if we show that $t \in P(S)$.
If (c.1) is satisfied, then $t \in W(S) = P(S)$.
If (c.2) is satisfied, then it follows from the fact that $a = \lim_{k\to\infty} a(n(k))$ and $\phi(a(n(k)), S_{n(k)}) \geqslant a(n(k))$ that $t \geqslant a \geqslant (\bar{p}_1(S), \underline{p}_2(S))$, which implies that $t \in P(S) = W(S) \cap \{x \geqslant \mathbf{R}^2; x_1 \geqslant \underline{p}_1(S), x_2 \geqslant \bar{p}_2(S)\}$.
If (c.3) is satisfied, then $\lim_{k\to\infty} \phi_1(a(n(k)), S_{n(k)}) \geqslant \lim_{k\to\infty} a_1(n(k)) = a_1 \geqslant \bar{p}_1(S)$. Hence, $\lim_{k\to\infty} \phi(a(n(k)), S_{n(k)}) \in P(S) \cup \underline{W}(S) = P(S)$.
Similarly, (c.4) implies that $\lim_{k\to\infty} \phi(a(n(k)), S_{n(k)}) \in P(S) \cup \bar{W}(S) = P(S)$.

(ii) To prove the other part of the theorem, we show that there exists an u.s.c. solution $\psi: B \to \mathbf{R}^2$, which is not continuous in (a, S) if (a, S) is a bargaining pair, which satisfies none of the conditions in the theorem.

Obviously, in that case, (a, S) satisfies one of the following two conditions

(D.1) $a_1 < \bar{p}_1(S)$ and $\bar{W}(S) \neq \{\bar{p}(S)\}$,

(D.2) $a_2 < \underline{p}_2(S)$ and $\underline{W}(S) \neq \{\underline{p}(S)\}$.

Suppose that (D.1) holds. We will prove that ψ^2 is not continuous in (a, S). Let w^* be the element in $\bar{W}(S)$ with minimal first coordinate. Let $\beta_n : \mathbf{R} \to \mathbf{R}$ be the function with

$$\beta_n(x) = \begin{cases} 0 & x \geqslant \bar{p}_1(S) \\ 1/n(\bar{p}_1(S) - x) \quad \text{if} & w_1^* \leqslant x < \bar{p}_1(S) \\ 1/n(\bar{p}_1(S) - w_1^*) \quad \text{if} & x < w_1^*. \end{cases}$$

Let $\lambda_n: \mathbf{R}^2 \to \mathbf{R}^2$ be the map with $\lambda_n(x_1, x_2) = (x_1, x_2 + \beta_n(x_1))$. For each $n \in \mathbf{N}$, let $(a(n), S_n) \in B$ be defined by $a(n) = \lambda_n(a)$ and $S_n = \text{conv}(\lambda_n(S))$. Then $\lim_{n\to\infty} d((a(n), S_n), (a, S)) = \lim_{n\to\infty} 1/n(\bar{p}_1(S) - w_1^*) = 0$. Furthermore, for all $n \in \mathbf{N}$, $\lambda_n(w^*) = \bar{p}(S_n)$. Now let $\epsilon = \min\{\bar{p}_1(S) - a_1, \bar{p}_1(S) - w_1^*\}$. Then the fact that, for all $n \in \mathbf{N}$,

$$\psi_1^2 (a (n), S_n) = \max \{a_1 (n), w_1^*\} = \max \{a_1, w_1^*\} \leq \bar{p}_1 (S) - \epsilon = \psi_1^2 (a, S) - \epsilon,$$

implies that $\lim_{n \to \infty} \psi^2 (a (n), S_n) \neq \psi^2 (a, S)$.

Similarly, if (D.2) holds, then ψ^1 is not continuous in (a, S). Q.E.D.

Remark. It is not difficult to show that the set of bargaining pairs (a, S) satisfying (c.1) is a dense subset of $\mathcal{B}$.

Now, we investigate the continuity of the coordinate functions $\phi_1 = \pi_1 \circ \phi$ and $\phi_2 = \pi_2 \circ \phi$ of an u.s.c. solution ϕ.
Let $\bar{\mathcal{B}} (\phi) = \{(a, S) \in \mathcal{B}; \phi (a, S) \neq \underline{p} (S)\}$, $\underline{\mathcal{B}} (\phi) = \{(a, S) \in \mathcal{B}; \phi (a, S) \neq \bar{p} (S)\}$ and $\mathcal{B} (\phi) = \bar{\mathcal{B}} (\phi) \cap \underline{\mathcal{B}} (\phi)$. The following holds.

Theorem 3.4. Let ϕ: $\mathcal{B} \to \mathbf{R}^2$ be an u.s.c. solution. Then

(1) *$\bar{\mathcal{B}} (\phi)$, $\underline{\mathcal{B}} (\phi)$ and $\mathcal{B} (\phi)$ are open subsets of $\mathcal{B}$.*

(2) *For each $(a, S) \in \bar{\mathcal{B}} (\phi)$, ϕ_2 is continuous in (a, S).*

(3) *For each $(a, S) \in \underline{\mathcal{B}} (\phi)$, ϕ_1 is continuous in (a, S).*

(4) *ϕ is continuous in each point $(a, S) \in \mathcal{B} (\phi)$.*

Proof. (a) Note, that $\bar{\mathcal{B}} (\phi) = \{(a, S) \in \mathcal{B}; \phi_1 (a, S) - \underline{p}_1 (S) < 0\}$.
Since ϕ_1 is u.s.c. and the map $(a, S) \to \underline{p}_1 (S)$ is continuous, we have that $(a, S) \to \phi_1 (a, S) - \underline{p}_1 (S)$ is u.s.c. Hence, $\bar{\mathcal{B}} (\phi)$ is an open subset of $\mathcal{B}$.
Similarly, it follows that $\underline{\mathcal{B}} (\phi)$ is open. But then also $\mathcal{B} (\phi)$ is an open set.
(b) Now we want to prove (2). Take $(a, S) \in \bar{\mathcal{B}} (\phi)$. Let $\langle (a (n), S_n); n \in \mathbf{N}\rangle$ be a sequence in $\mathcal{B}$, converging to (a, S). Let $(a (n (1)), S_{n(1)}), (a (n (2)), S_{n(2)}), \ldots$ be a subsequence for which $z = \lim_{k \to \infty} \phi (a (n (k)), S_{n(k)})$ exists. By theorem 3.2, we obtain $z \in P (S) \cup \bar{W} (S)$. Now, for each $w \in P (S) \cup \bar{W} (S)$, we have $\pi_2 \circ j_S (w) = \pi_2 (w)$. This implies, in view of theorem 3.2, that

$$\phi_2 (a, S) = \pi_2 \circ \phi (a, S) = \pi_2 \circ j_S (\lim_{k \to \infty} \phi (a (n (k)), S_{n(k)}) = \pi_2 (z).$$

So we may conclude that $\langle \phi_2 (a (n), S_n); n \in \mathbf{N}\rangle$ converges to $\phi_2 (a, S)$.
(c) Similarly, as in (b), one can prove assertion (3). Assertion (4) is a straightforward consequence of (2) and (3). Q.E.D.

4. Lower Semicontinuous Weak Solutions

Another approach to the bargaining problem is the construction of functions ϕ: $\mathcal{B} \to \mathbf{R}^2$, satisfying

(W.1) $\phi(a, S) \in W(S)$, for each $(a, S) \in B$ *(Weak Pareto optimality)*,

(W.2) $\phi(a, S) \geqslant a$, for each $(a, S) \in B$ *(Individual rationality)*,

$\phi(a, S) = a$, for each $(a, S) \in B$ with $a \in W(S)$.

Such functions are called *weak solutions*. There does not exist an u.s.c. weak solution as we will show now.

For each $n \in \mathbf{N}$, let $a(n) = (-1/n, 0)$, $S_n = [(-1/n, 0), (0, 1)]$, $a = (0, 0)$ and $S = [(0, 0), (0, 1)]$. For each weak solution $\phi: B \to \mathbf{R}^2$, we have $\lim_{n\to\infty} \phi(a(n), S_n) = (0, 1)$ and $\phi(a, S) = (0, 0)$. Hence, ϕ is not u.s.c.

A weak solution is called *lower semicontinuous* (l.s.c.) if
(W.3) for each sequence $(a, S), (a(1), S_1), (a(2), S_2), \ldots$, in B with

$$\lim_{n\to\infty} (a(n), S_n) = (a, S), \text{ we have}$$

$$\phi_i(a, S) \leqslant \liminf_{n\to\infty} \phi_i(a(n), S_n), \text{ for each } i \in \{1, 2\}.$$

Example. For $i \in \{1, 2\}$ and each $(a, S) \in B$, let $\omega^i(a, S)$ be equal to a, if $a \in W(S)$; if $a \notin W(S)$, then let $\omega^1(a, S)$ $(\omega^2(a, S))$ be the element of $\{w \in W(S); w \geqslant a\}$ with smallest second (first) coordinate. The maps $\omega^1: B \to \mathbf{R}^2$ and $\omega^2: B \to \mathbf{R}^2$ turn out to be l.s.c. weak solutions. We only prove that ω^2 is l.s.c.

That ω_1^2 is l.s.c., is due to the fact that

$$\omega_1^2(a, S) = \min \{w_1; w \in W(S), w \geqslant a\}$$

and that the multifunction $(a, S) \to \{w_1; w \in W(S), w \geqslant a\}$ is closed.

Concerning ω_2^2, note that

$$\omega_2^2(a, S) = \begin{cases} \max \{p_2; p \in P(S), p \geqslant a\} & \text{if } a \notin \bar{W}(S) \\ a_2 & \text{if } a \in \bar{W}(S). \end{cases}$$

That ω_2^2 is l.s.c. in points (a, S) with $a \notin \bar{W}(S)$, follows from the fact that the multifunction

$$(a, S) \to \{p_2; p \in P(S), p \geqslant a\}$$

is l.s.c. For an $(a, S) \in B$ with $a \in \bar{W}(S)$ and sequences $\langle (a(n), S_n); n \in \mathbf{N} \rangle$ in B, converging to (a, S), we have

$$\liminf_{n\to\infty} \omega_2^2(a(n), S_n) \geqslant \liminf_{n\to\infty} a_2(n) = a_2 = \omega_2^2(a, S).$$

Hence, also ω_2^2 is l.s.c.

Theorem 4.1. Let ϕ: $\mathcal{B} \to \mathbf{R}^2$ be a weak solution. Then ϕ is l.s.c. in each point $(a, S) \in \mathcal{B} \setminus \mathcal{B}^0$.

Proof. Suppose that ϕ is not l.s.c. in $(a, S) \in \mathcal{B} \setminus \mathcal{B}^0$. Then there exists a sequence $(a(1), S_1), (a(2), S_2), \ldots$ in $\mathcal{B}$ converging to (a, S) such that $t := \lim_{n\to\infty} \phi(a(n), S_n)$ exists but $t \geqslant \phi(a, S)$ does not hold. Suppose that $t_1 < \phi_1(a, S)$. Then, by (W.2), $t_1 < \phi_1(a, S) = a_1$. Now $\phi_1(a(n), S_n) \geqslant a(n)_1$ for all n. Taking limits we obtain $t_1 = \lim_{n\to\infty} \phi_1(a(n), S_n) \geqslant \lim_{n\to\infty} a(n)_1 = a_1 > t_1$ which is impossible. Similarly, $t_2 < \phi_2(a, S)$ leads to a contradiction. Q.E.D.

The proofs of the following two theorems are left to reader.

Theorem 4.2. Let ϕ be a l.s.c. weak solution. Suppose that, for a sequence $(a, S), (a(1), S_1), (a(2), S_2), \ldots$ in $\mathcal{B}$, $\lim_{n\to\infty} (a(n), S_n) = (a, S)$ and $\lim_{n\to\infty} \phi(a(n), S_n)$ exists. Then we have:

(i) *If $\lim_{n\to\infty} \phi(a(n), S_n) \in \mathcal{P}(S) \setminus \{\bar{p}(S), \underline{p}(S)\}$, then $\phi(a, S) = \lim_{n\to\infty} \phi(a(n), S_n)$.*

(ii) *If $\lim_{n\to\infty} \phi(a(n), S_n) \in \bar{\mathcal{W}}(S)$, then $\phi(a, S) \in [\bar{w}(S), \lim_{n\to\infty} \phi(a(n), S_n)]$,*

where $\bar{w}(S) = (\min_{w\in\mathcal{W}(S)} w_1, \max_{w\in\mathcal{W}(S)} w_2)$.

(iii) *If $\lim_{n\to\infty} \phi(a(n), S_n) \in \underline{\mathcal{W}}(S)$, then $\phi(a, S) \in [\underline{w}(S), \lim_{n\to\infty} \phi(a(n), S_n)]$,*

where $\underline{w}(S) = (\max_{w\in\mathcal{W}(S)} w_1, \min_{w\in\mathcal{W}(S)} w_2)$.

Theorem 4.3. Let ϕ be a l.s.c. weak solution. Let $(a(1), S_1), (a(2), S_2), \ldots$ be a sequence in $\mathcal{B}$, converging to (a, S). *Then*

$$\lim_{n\to\infty} \phi(a(n), S_n) = \phi(a, S), \textit{ if } \phi(a, S) \in \mathcal{P}(S).$$

Example. Let $S = \text{conv}\{(0, 0), (1, 0), (0, 1), (1, 1)\}$. Let $a = (0, 1)$ and let, for each $n \in \mathbf{N}$, $a(n) = (0, 1 - n^{-1})$. Let ω^1: $\mathcal{B} \to \mathbf{R}^2$ be the l.s.c. weak solution, introduced before. Note, that

$$\lim_{n\to\infty} \omega^1(a(n), S) = \lim_{n\to\infty} (1, 1 - n^{-1}) = (1, 1) \neq (0, 1) = \omega^1(a, S).$$

Hence, the condition that $\phi(a, S) \in \mathcal{P}(S)$ in theorem 4.3, is not superfluous.

Theorem 4.4. A l.s.c. weak solution is continuous in (a, S) if and only if one of the four conditions, mentioned in theorem 3.3, holds.

Proof. We only show that there exists a l.s.c. weak solution ω: $\mathcal{B} \to \mathbf{R}^2$ which is not continuous in (a, S) if (a, S) satisfies none of the conditions mentioned in theorem 3.3. Obviously, in that case, (a, S) satisfies one of the conditions (D.1) or (D.2), as formulated in the proof of theorem 3.3.

If (D.1) holds, take $\omega = \omega^2$. Then $\lim_{n \to \infty} \omega^2 (a(n), S_n) \neq \omega^2 (a, S)$ if $a(n) = \mu_n(a)$ and $S_n = \operatorname{conv}(\mu_n(S))$, where

$$\mu_n(x_1, x_2) = (x_1, x_2 - \beta_n(x_1)), \quad \text{for each } (x_1, x_2) \in \mathbf{R}^2,$$

and β_n is the function, introduced in the proof of theorem 3.3.

If (D.2) holds, then ω^1 is not continuous in (a, S). Q.E.D.

Theorem 4.5. Let ϕ: $\mathcal{B} \to \mathbf{R}^2$ be a l.s.c. weak solution. Let $\overline{\mathcal{B}}^w(\phi) = \{(a, S) \in \mathcal{B}; \phi(a, S) \notin \underline{W}(S)\}$, $\underline{\mathcal{B}}^w(\phi) = \{(a, S) \in \mathcal{B}; \phi(a, S) \notin \overline{W}(S)\}$ and $\mathcal{B}^w(\phi) = \overline{\mathcal{B}}^w(\phi) \cap \underline{\mathcal{B}}^w(\phi)$. Then $\overline{\mathcal{B}}^w(\phi)$, $\underline{\mathcal{B}}^w(\phi)$ and $\mathcal{B}^w(\phi)$ are open subsets of $\mathcal{B}$. Furthermore,

(1) *ϕ_2 is continuous in each point $(a, S) \in \overline{\mathcal{B}}^w(\phi)$,*

(2) *ϕ_1 is continuous in each point $(a, S) \in \underline{\mathcal{B}}^w(\phi)$,*

(3) *ϕ is continuous in each point $(a, S) \in \mathcal{B}^w(\phi)$.*

Proof. The set $\overline{\mathcal{B}}^w(\phi) = \{(a, S) \in \mathcal{B}; \phi_2(a, S)\ \underline{p}_2(S) > 0\}$ is open, because $(a, S) \to \phi_2(a, S) - \underline{p}_2(S)$ is a l.s.c. function on $\mathcal{B}$. Similarly, $\underline{\mathcal{B}}^w(\phi)$ is open. Then also $\mathcal{B}^w(\phi)$ is open. Now we want to prove (1). Take $(a, S) \in \overline{\mathcal{B}}^w(\phi)$. Let $\langle (a(n), S_n); n \in \mathbf{N} \rangle$ be a sequence in $\mathcal{B}$, converging to (a, S). Let $\langle (a(n(k)), S_{n(k)}); k \in \mathbf{N} \rangle$ be a subsequence with $\lim_{k \to \infty} \phi(a(n(k)), S_{n(k)}) = z$. By theorem 4.2 (iii), $z \notin \underline{W}(S)$, because $\phi(a, S) \notin \underline{W}(S)$. If $z \in P(S) \setminus \{\bar{p}(S)\}$, then $z = \phi(a, S)$, by theorem 4.2 (i). If $z \in \overline{W}(S)$, then $z_2 = \phi_2(a, S)$, by theorem 4.2 (ii). Hence, $\lim_{k \to \infty} \phi_2(a(n(k)), S_{n(k)}) = \phi_2(a, S)$ in both cases. This implies that ϕ_2 is continuous on $\overline{\mathcal{B}}^w(\phi)$.

Similarly, one proves assertion (2) of the theorem and assertion (3) follows immediately from (1) and (2). Q.E.D.

5. Non-Compact Bargaining Pairs

In the foregoing, we restricted our attention to bargaining pairs (a, S), where S is compact. Many results can be extended to non-compact bargaining pairs, if we concentrate our attention on an appropriate family $\mathcal{B}^*$ of bargaining pairs and if we topologize $\mathcal{B}^*$ carefully. In the following, $\mathcal{K}^*$ is the family of closed convex subsets of $\mathbf{R}^2$ with non-empty Pareto set. We will say that a sequence $S_1, S_2, \ldots$ in $\mathcal{K}^*$ converges to $S \in \mathcal{K}^*$, if $\liminf_{n\to\infty} S_n \supset S \supset \limsup_{n\to\infty} S_n$ [cf. *Salinetti/Wets*]. Now $\mathcal{B}^* = \{(a, S); a \in S \text{ and } S \in \mathcal{K}^*\}$ is called the set of *closed bargaining pairs.* Note, that $\mathcal{B} \subset \mathcal{B}^*$. We will say that the sequence $\langle (a\,(n), S_n); n \in \mathbf{N}\rangle$ in $\mathcal{B}^*$ converges to $(a, S) \in \mathcal{B}^*$, if $\lim_{n\to\infty} \| a\,(n) - a \|_\infty = 0$ and $\lim_{n\to\infty} S_n = S$.

A solution of the bargaining problem is now a map $\phi\colon \mathcal{B}^* \to \mathbf{R}^2$ with $\phi\,(a, S) \in P\,(S)$ and $\phi\,(a, S) \geqslant a$, for each $(a, S) \in \mathcal{B}^*$. It is straightforward to generalize almost all lemmas, propositions and theorems in section 3 (an exception is proposition 3.1 (b) because ϕ^{KR} is not defined on $\mathcal{B}^*$) by modifying the proofs. Essential is the following lemma, which we give without proof.

Lemma 5.1. Let $(a\,(1), S_1), (a\,(2), S_2), \ldots$ be a sequence in $\mathcal{B}^*$ converging to (a, S). Then $\langle \phi\,(a\,(n), S_n); n \in \mathbf{N}\rangle$ is a bounded sequence.

Also many of the results in section 4 for weak solutions can be extended to weak solutions $\phi\colon \mathcal{B}^* \to \mathbf{R}^2$.

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